THE PLAYGROUND!

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Welcome to the Playground! Playground rules are posted at the end of page 33, except for the most important one: *Have fun*!

THE SANDBOX

In this section, we highlight problems that anyone can play with, regardless of mathematical background. But just because these problems are easy to approach doesn't necessarily mean that they are easy to solve!

The points on the real number line $\boldsymbol{R}$ have the special property of being *totally ordered*: any two real numbers a and b are related either by $a \le b$ or $b \le a$. (Many thanks to the 1-dimensional line for making this possible!) For the Sandbox problem, we consider a slightly different type of order, called a *partial order*, for points in 2-dimensional and 3-dimensional space since these points don't lie naturally on a 1-dimensional line.

For two points (a,b) and (c,d) in the 2-dimensional plane $\boldsymbol{R}^2$, we'll say that $(a,b) \le (c,d)$ if both $a \le c$ and $b \le d$ are true for the real numbers a, b, c, and d. For example, $(1,5) \le (3,11)$ and $(-1,-2) \le (5,-2)$; but it is not true that $(1,\pi) \le (203,3)$ because it is not true that $\pi \le 3$. Nor is it true that $(203,3) \le (1, \pi)$, which means that only some pairs of points in $\boldsymbol{R}^2$ are related by this new notion of "less than or equal to."

In an analogous "coordinate-wise" way we can define $\le$ for points in $\boldsymbol{R}^3$: we'll say that $(a,b,c) \le (d,e,f)$ if $a \le d$, $b \le e$, and $c \le f$.

As a way to first start thinking about this new type of order, try to find three points in $\boldsymbol{R}^2$ such that no pair of the points is related by $\le$. An example of such points is given later in the section "Cleaning Up."

Problem 232, Orderly Points, is offered by Gary Gordon of Lafayette College. Suppose that you are given a set of infinitely-many distinct points (x,y) in $\boldsymbol{R}^2$, where all of the x and y values are positive integers.

(a) Show that there must be a pair of distinct points (x_1, y_1) and (x_2, y_2) such that $(x_1, y_1) \le (x_2, y_2)$.

(b) Show that in fact there must be a chain of infinitely-many such distinct points:

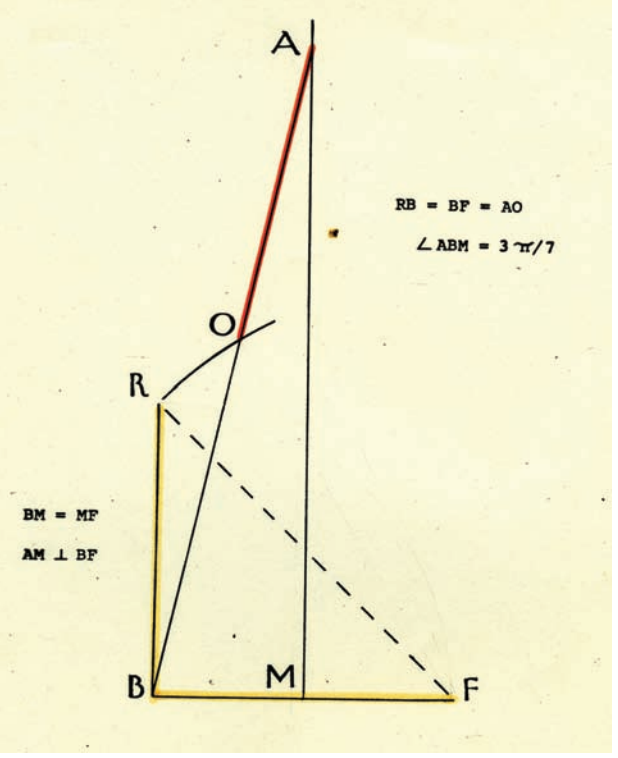
$$(x_1, y_1) \le (x_2, y_2) \le (x_3, y_3) \le (x_4, y_4) \le \ldots$$

(c) What happens in $\boldsymbol{R}^3$? If you are given a set of infinitely-many distinct points (x,y,z) in $\boldsymbol{R}^3$, where x, y, and z are positive integers, must there be two distinct points related by $\le$? An infinite chain of them?

THE ZIP-LINE

This section offers problems with connections to articles that appear in this issue. Not all of the problems in this section require you to read the corresponding articles, but doing so can never hurt, of course.

The following figure is copied from the article "Harold and the Purple Heptagon" on pages 5-9. Line segments RB and BF form two sides of a square, and the curve drawn through points O and R is an arc of a circle centered at F. AM is a perpendicular bisector of BF, and AO has the same length as BF. **Problem 233, A Slice of Purple Pi,** is to show that $\angle ABF = (3/7)\,\pi$.



One way to begin thinking about the problem is to ask yourself how you might prove, in any situation, that a given angle has measure of the form $(k/7)\,\pi$ for an integer k.

THE JUNGLE GYM

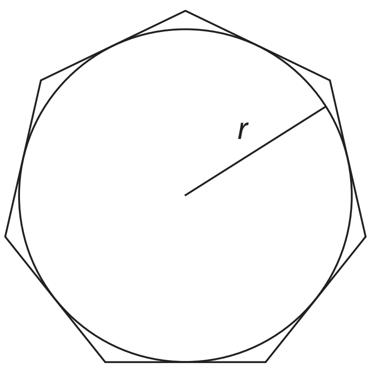
Any type of problem may appear in the Jungle Gym—climb on!

David Atkinson of Olivet Nazarene University offers **Birthday Cruise,** a problem that occurred to him when he and his wife took a trip to Eastern Europe that included a 7-day cruise on the Danube River. After dinner on the first evening of the cruise the tour leaders and some of the kitchen crew paraded into the dining room with a small cake singing "Happy Birthday!" to a passenger whose birthday was that day. Professor Atkinson immediately wondered what the chances were for one or more matching birthdays (same month and day, but not necessarily the same year) during the cruise among the 160 passengers, as this event would require the crew to prepare two or more cakes on the same day.

Problem 234 is to determine whether a birthday match is more likely than not to occur in a group of 160 people during a 7-day period.

A second Jungle Gym problem, **Derived Perimeters,** is offered by Nathan Carter of Bentley University, who was asked by a curious student if the following fact was a coincidence: the derivative of the area of a circle, πr^2, taken with respect to its radius r, is equal to its perimeter, $2\pi r$. This led to a discussion by Nathan and his faculty colleagues, who noted also that the derivative of the volume of a sphere, $(4/3)\pi r^3$, is equal to its surface area, $4\pi r^2$.

The conversation turned to consideration of 2-dimensional figures other than the circle: do any have such a nice relationship between area and perimeter? Here's a partial answer, in the form of **Problem 235:** show that the derivative of the area of a regular n-gon, taken with respect to the radius of the inscribed circle, is equal to the n-gon's perimeter. A figure for $n = 7$ is given below.



Results for other 2- and 3-dimensional figures are also encouraged, including the following two specific challenges. What's the largest class of 2-dimensional figures you can find for which the derivative of area is perimeter? And do any of the five Platonic solids have the derivative of volume equal to surface area?

FEBRUARY WRAP-UP

February's Playground led off with a chip sorting problem from Steven Tedford of Misericordia University. There are initially $n+1$ red chips alternating positions in a row with n blue chips. The row of chips may be altered by the following type of move: any two consecutive chips may be moved to either end of the row, with the gap they left behind squeezing shut. **Problem 224,** the **Squeezed Chip Puzzle,** asked three questions about $s(n)$, the least number of moves required to produce a row with all of the red chips to the left of all of the blue chips:

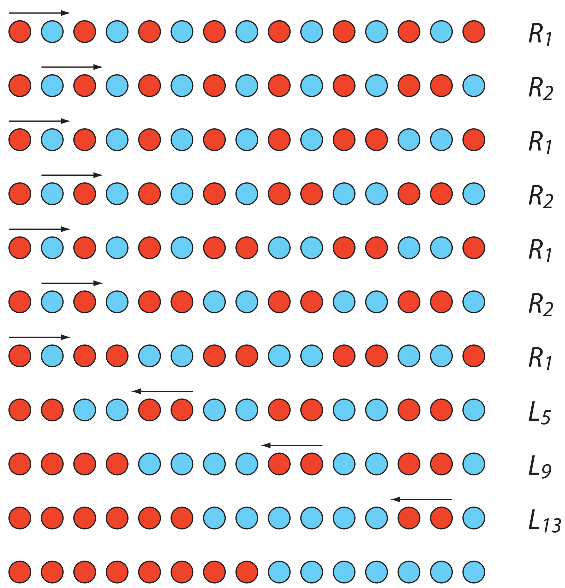
(a) Is $s(7) \leq 10$?

(b) Can you find an upper bound function $g(n)$ such that $s(n) \leq g(n)$ for all n?

(c) Can you find a simple formula for $s(n)$?

The correct answer to (a) was given by students Tim Hayes (Messiah College), Michael Abram (University of the Pacific), Junjie Wu and Shihan Qin (University of Michigan & Shanghai Jiao Tong University Joint Institute), the Armstrong Problem Solvers, and the Bethel College Problem-Solving Group, as well as by student Ethan Charles Blocher-Smith (Huntington University) and two student groups from Taylor University. The group from Bethel College (Mishawaka, IN) consists of students Daniel Hill, Eric Hooley, Ronald Powell, Chad Stephenson, and Seth Thompson, with Professor Adam Hammert. One group from Taylor University (Upland, IN) consists of students David Ebert, Jacob Erb, Daniel Kasper, and Joseph Lupton ("Taylor A"), while a second group consists of students Amanda Currie, Rebecca Livingston, Emily Watkins, and Brittany Wood ("Taylor B").

Each solution to part (a) described a specific sequence of 10 moves that achieves the desired sort, which implies that the minimum number of moves $s(7)$ can be no greater than 10. Shihan's sequence is given below, where we introduce notation for the moves: L_i is the move that takes the chips in positions i and $i+1$ to the left end of the row, while R_i takes them to the right end.



The first six solvers listed above for (a) also submitted correct solutions for part (b), each describing a specific sequence of moves for sorting the chips that breaks into two cases, depending on whether n is even or odd. All of the solutions presented were variants of the following method, modeled on the solution from the group from Armstrong Atlantic and generalizing the example presented above in part (a). Repeat the moves R_1 and R_2 a total of $\lfloor (n-1)/2 \rfloor$ times, followed by one more R_1 move (where $\lfloor x \rfloor$ is the "floor" of x, the greatest integer less than or equal to x). At this point, all of the red chips will be paired together, with the exception of one red chip on the left end of the row in the case when n is even. Regardless of the parity of n, you can check that then $\lfloor n/2 \rfloor$ pairs of red chips may be moved to the left end of the row to complete the sort, for a grand total of

$$2\lfloor (n-1)/2 \rfloor + 1 + \lfloor n/2 \rfloor$$

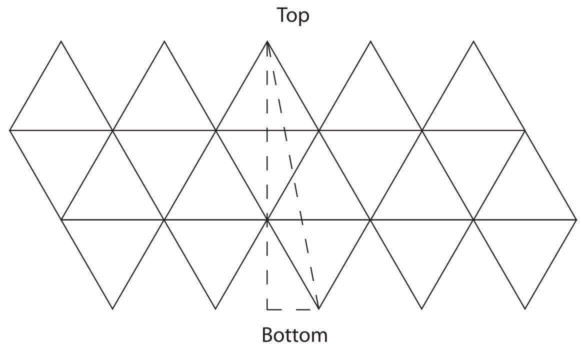
moves. You can also check that this last expression is in fact equivalent to $\lfloor (3n-1)/2 \rfloor$ by considering its value when n is either even or odd.

This shows that $s(n)$ is bounded above by $g(n) = \lfloor (3n-1)/2 \rfloor$ for all n, answering the question in part (b). For part (c), several submissions attempted to show that $s(n) = g(n)$ by explaining why any possible sorting method would require at least $g(n)$ moves. None of the proposed solutions were completely successful, so we'll keep this part of the problem open.

An Icosahedral Climb described the situation of an adventurous ant approaching an icosahedron that is balancing on one of its vertices. **Problem 225** asked you to help the ant find the path of shortest distance to the top of the icosahedron.

One obvious way to the top follows a zig-zag-zig path along 3 of the 30 edges of the icosahedron. If the length of one edge is s, then the total length of this path is $3s$. A more direct path first sends the ant up an altitude of one triangle, then up a second altitude, and finally along an edge to the top, a path that has a total length of $(1+\sqrt{3})s$.

However, the ant can do better, as described in solutions from students Tim Hayes and Michael Abram, and the Armstrong Problem Solvers, the Bethel College Problem-Solving Group, and Taylor A. The key is to "unfold" the icosahedron, such as in the figure below, and then determine the shortest Euclidean distance from any point corresponding to the bottom vertex to any point corresponding to the top vertex.



Tim explains how the length of the slanted dotted path shown above can be computed using the Pythagorean Theorem: since the short dotted leg has length $s/2$ and the long dotted leg has the length of three triangle altitudes, the total path length is

$$\sqrt{(s/2)^2 + (3\sqrt{3}s/2)^2} = s\sqrt{7}.$$

But we were asked to help the ant, not just compute the length of the shortest path! The Armstrong Problem Solvers give the ant explicit instructions:

> From the bottom vertex, crawl in a straight line along any of the lower five faces to a point 1/3 of the way along the opposite edge of that face. On the next face, crawl in a straight line to the midpoint of the nearer of the other two edges in that face. Crawl up the next face to a point 1/3 of the way along the horizontal edge of that face. Finally, crawl on the next face to the opposite vertex, which will be the top.

The group from Bethel College challenges you to solve the analogous problem for the cube, octahedron, and dodecahedron. (For an action-packed image of someone considering the problem on the cube, look no further than page 21 of last November's *Horizons*.) For another challenge of this type, find one of the many online references to Henry Dudeney's classic problem of the spider and the fly.

Rational Trigonometric Sum was a Zip-Line problem linked to the article "Values of Trigonometric Functions" by Jeffrey Bergen in the February issue. The article gave a characterization by elementary methods of all rational angles α (angles that are π times a rational number) such that both $\sin^2(\alpha)$ and $\cos^2(\alpha)$ are rational numbers. **Problem 226** asked which rational angles α make the single quantity $\sin(\alpha) + \cos(\alpha)$ a rational number.

Correct solutions were given by students Michael Abram and Santhosh Karnik (Wheeler High School, GA), as well as by the Armstrong Problem Solvers and the Northwestern University Problem Solving Group. There was one incorrect solution. Three of the solutions used the fact that $(\sin(\alpha) + \cos(\alpha))^2 = 1 + \sin(2\alpha)$. As the group from Northwestern explains, if $\sin(\alpha) + \cos(\alpha)$ is rational, so is its square $(\sin(\alpha) + \cos(\alpha))^2$, and so $\sin(2\alpha)$ must be rational. By the first theorem in Bergen's article, this is enough to limit the value of $\sin(2\alpha)$ to one of the values –1, –1/2, 0, 1/2, and 1. The values –1/2, 1/2, and 1 can be ruled out because they make $\sqrt{(1 + \sin(2\alpha))}$ irrational. The only permissible values of $\sin(2\alpha)$ are 0 and –1, which occur just when $\alpha = \pi k/2$ and $\alpha = 3\pi/4 + \pi k$ for any integer k.

"FIVE MORE MINUTES, KIDS!"

In **Glimpsing a Heart,** Jeffrey Liebner posed a probability question. You are playing a game where two cards are dealt face down to each player. As the cards are being dealt and before you see your own cards, you catch a glimpse of the cards of the opponent sitting to your right: all you can conclude is that at least one of the cards is a heart. **Problem 227** asked whether that glimpse changes the probability that your opponent has two aces.

The Armstrong Problem Solvers, the Bethel College Problem-Solving Group, Taylor A, the Skidmore College Problem Group, and student David Montgomery (Westmont College) all arrived at the same two probabilities and the conclusion that, yes, the probability does change. A variety of methods were used to compute the two probabilities; what follows is David's approach. Before the glimpse, since there are $C(4,2) = 6$ unordered pairs of aces out of $C(52,2) = 1326$ possible pairs of cards, the probability that the opponent has two aces is 6/1326 = 1/221. (Here $C(4,2)$ represents four cards taken two at a time, or "4 choose 2.") After the glimpse, only 3 of the original 6 pairs of two aces are possible. Also, the total number of possible pairs is reduced by $C(39, 2) = 741$, the number of pairs containing no heart, leaving 1326–741 = 585 pairs with at least one heart. This gives a new probability of 3/585 = 1/195, which is greater than 1/221.

The Skidmore group approached the problem using conditional probability. If A is the event that the opponent has two aces and H is the event that the opponent has at least one heart, then the problem is asking if $P(A)$ and $P(A|H)$ differ. The latter quantity is a conditional probability, the probability that A occurs given that H occurs; it can be computed using the formula $P(A|H) = P(A \cap H) / P(H)$, where $A \cap H$ is the event that the opponent has both two aces and at least one heart. After computing the probabilities $P(A \cap H)=1/442$ and $P(H)=15/34$, the group also arrived at $P(A)=1/221$ and $P(A|H)=1/195$.

However, all of the solutions made one implicit assumption: that all pairs of cards containing at least one heart are *equally likely* to be glimpsed. Given the admittedly imprecise statement of the problem, that might not be the case. Suppose, for instance, that you have a better chance of identifying the suit of the *first* of the two cards being dealt to your opponent, a reasonable scenario if your opponent is taking a quick single peek at both of her cards together on the table. One version of this situation would have a heart identifiable during the glimpse if and only if it appeared on the first of the two cards dealt. In this case, if you recompute $P(A \cap H)$ and $P(H)$ above you might just find that the probabilities $P(A)$ and $P(A|H)$ don't differ by much, if at all!

CLEANING UP

Here's an example of three points in $\mathbf{R}^2$ such that no pair of the points is related by ≤: (1,4), (2,3), and (3,2). Good luck with Problem 232!

Also, Michael Abram correctly solved Problem 222, but his solution was not acknowledged in the April issue.

SUBMISSION & CONTACT INFORMATION

The Playground features problems for students at the undergraduate and (challenging) high school levels. All problems and/or solutions may be submitted to Derek Smith, Mathematics Department, Lafayette College, Easton, PA 18042. Electronic submissions (preferred) may also be sent to `smithder@lafayette.edu`. Please include your name, email address, school affiliation, and indicate if you are a student. If a problem has multiple parts, solutions for individual parts will be accepted. Unless otherwise stated, problems have been solved by their proposers.

The deadline for submitting solutions to problems in this issue is November 10, 2009.

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