The Longest Day

The Earth spins with axis and rate of rotation which can change only due either to precession as a result of being in a gravitational gradient (which causes the axis to wobble slightly with a period of about 26000 years), or to transfer of angular momentum due to tidal interactions with the moon and other bodies (which extends the spin period by about 2.3 milliseconds per century). Thus the "sidereal day" length (defined as the time required for one complete revolution relative to an inertial system – or equivalently to the "fixed" stars) is essentially constant from the point of view of an average human observer (or anyone measuring a single day length without an atomic clock). Because the Earth orbits around the sun once a year, the apparent position of the sun among the stars advances by about 1/365 of a revolution every day and so from noon to noon the Earth has to rotate a bit more than a full revolution (it would be correspondingly less if the directions of rotation were opposite as on a 'retrograde' planet, but in fact they are the same – ie the Earth is 'prograde'). Thus the "solar day" of 24 hours is about 0.3% longer than the sidereal day – with further slight variations discussed below.

Seasons arise as a result of fact that the Earth's equator is not in the same plane as its orbit (or equivalently that its axis of spin is not perpendicular to the orbital plane). If this were not the case (ie if the spin axis was perpendicular to the orbital plane), then the angle of elevation of the sun at noon would always be equal to the complement of the latitude of the observer, but because of the axial tilt (which is about $23°$) the sun's elevation at noon varies over the year between latitude plus tilt angle (which happens when our hemisphere is tilted towards the sun in our summer) and latitude minus tilt angle (which happens in winter). The extremes are called 'solstices' (occurring at June 20-21 and Dec 21-22) and the times when the sun appears in the plane of the equator are the 'equinoxes'.

If the sun's apparent angular velocity around the Earth's axis is constant, then the day of longest sun must occur when the highest proportion of the solar path is above the horizon – which is exactly the Summer solstice. But if that angular velocity varies over the year, then it may be that variations of solar angular velocity move the day of longest sunlight to a slightly different date. It is the purpose of this document to check whether that is indeed the case.

First let us consider how big a change would be required to move the longest day.
Any phenomenon that is going to change the date of the longest sunlit day is going to have to have an effect which is bigger than that of the change in solar elevation between the solstice and the next or previous day. This may not be too hard though, since the rate of change of solar elevation is zero at the moment of solstice and so the difference between that and the next or prior day is proportional to the square rather than the first power of the time difference.

In fact, in middle latitudes the proportion of the day during which the sun is above the horizon oscillates according to the formula $F \approx \frac{1}{2} + A\cos\left(\frac{2\pi}{365}t\right)$ where cos(x) is the cosine of an angle of x radians (for degrees we'd replace 2π with $360°$), *t* is the number of days away from the summer solstice, and the amplitude *A* varies with latitude.
(In Vancouver, *A* is about $\frac{1}{6}$ since day length varies between about 8 hours min to 16 hours max). A derivation of the exact formula is appended at the end of this note.

Since $\cos(x) \approx 1 - x^2/2$ for small x, near the solstice the change in sunlit fraction of the day over the next (or previous) t days is about $\frac{A}{2}\left(\frac{2\pi}{365}t\right)^2$ of a full day (or about $12At^2$ seconds). For us in Vancouver, this corresponds to a change of about 2 seconds for the first day, 8 sec for two days, about 2 minutes for a week and about half an hour for a month.

Now let's look at what other factors might change the time of daylight.
First, the apparent angular travel of the sun around the earth's spin axis due to orbital motion of the Earth is not quite constant for two main reasons. One is the ecliptic tilt and the other is the eccentricity of the orbit combined with the physics of orbital motion.

The eccentricity is 0.0167, so our distance from the sun only varies by $< \pm 2\%$ and so our orbital angular velocity (proportional by Kepler's 2nd law to R^{-2}) only varies by $\pm 4\%$. Since the orbital motion contribution to day length is in total only $\frac{1}{365}$ of a day, the total variation of day length due eccentricity is only about $\pm 4\%$ of $\frac{1}{365}$ of a day – ie about .0001 day or $\pm$ 9seconds for the full "24 hour" day and $\pm$ 4.5sec for the sunlit part. Spreading this over approximately 90 days for each quarter of the cycle gives an average daily change of 0.05 seconds of sunlight, with a maximum which may be a bit more. (Readers who know some calculus will see that $E(t) = 4.5\cos(2\pi t/360)$, gives $E'_{\max} = 4.5\left(2\pi/360\right) \approx 0.08$) This is nowhere near sufficient to overcome the change of daylight length on even the first day away from the solstice (and that's without taking account of the fact that the solstices are now close to perihelion and aphelion - where in fact the orbital angular velocity is at its extremes and so is barely changing from day to day).

Since the eccentricity is so small, the orbit is nearly circular and the effect of the tilt can be estimated assuming a circular orbit. The tilt effect is due to the fact that even if the orbital angular velocity were constant the amount of catch-up rotation needed to extend a sidereal to a solar day depends on the time of year. At the equinoxes the daily solar travel arc of length $\frac{2\pi}{365}R$ is crossing the equatorial plane at an angle of about $23°$ so its projection on the equatorial plane has length only $\frac{2\pi}{365}R\cos(23°)$ so the axial rotation needed to match this is only $\cos(23°)$ times $\frac{1}{365}$ of a revolution and so the solar day at either equinox is only $\cos(23°)$ times $\frac{1}{365}$ of a day longer than the fixed sidereal day. Or in other words the solar day at equinox is shorter than average by about $\frac{1-\cos(23°)}{365}$ of a day or about 20 seconds (or just 10 sec for the daylight half). On the other hand, at the solstices the daily solar travel arc is parallel to the equatorial plane but about $23°$ above or below it. So now projecting down shrinks the radius and the corresponding axial angle is increased by a factor of $\sec(23°)$. This has the effect of extending the full day by approximately 20 seconds (and so giving 10 sec. more daylight) at both solstices. In between solstices and equinoxes the tilt effect on solar day length oscillates with a period of six months, but the most extreme extension is at the solstices - so this effect just contributes more to the dominance of the summer solstice in terms of hours of daylight.

At the winter solstice this tilt effect does lengthen the day a bit, but if all nearby days are lengthened by the same amount then it won't change the date of the shortest day, and since the effect is maximum at the solstice the change from day to day is, at that point very small and much less than the corresponding change due to the primary seasonal effect. (In fact the change in amount of lengthening is equal to half the total amplitude of variation times the square of the time from solstice as a proportion of the cycle. So to make the neighbouring days actually have less hours of daylight than the solstice, a day-lengthening effect that was extreme at the solstice with half year period would have to have an amplitude more than 25% of the main effect and ours is over a hundred times less than that.)

Although they do not significantly affect the number of hours of daylight on any given day, what these effects *do* achieve is to effectively push the time of noon as measured by a steadily running clock forward and back from the time of solar noon, and this cumulative effect is quite noticeable. So the clock time at which the sun is highest oscillates back and forward around 12:00 with an amplitude which accumulates to about 15min either way (very roughly ¼ year of extensions which average about 10sec/day for the larger effect). And the clock times of sunrise and sunset are also shifted correspondingly. So in fact, although the solstice has the longest period of daylight, it has neither the earliest dawn nor the latest sunset.

The "Equation of Time"

The actual value of the noontime shift is called the "Equation of Time" (this being an antique use of the word "equation" to represent what is needed to make things equal rather than a statement of equality). For those who know calculus it can be calculated from our above results quite accurately by integrating.

For the eccentricity contribution we had day length adjustment of $9\cos({}^{2\pi t}\!/_{365})$ for the full 24 hour day where *t* is the time in days from perihelion (which is just 2 weeks after to the northern winter solstice). So the accumulated delay at *t* days after the solstice is $\int 9\cos({}^{2\pi(t-14)}\!/_{365})dt = 9\left({}^{365}\!/_{2\pi}\right)\sin({}^{2\pi(t-14)}\!/_{365})$ seconds or about $8.5\sin({}^{2\pi(t-14)}\!/_{365})$ minutes.

Using the more accurate figure of 0.0167 for the eccentricity would reduce this by about one sixth – to give a contribution of $7.3\sin({}^{2\pi(t-14)}\!/_{365})$

And for the tilt effect the day length adjustment was $20\cos({}^{4\pi t}\!/_{365})$ for an accumulated delay of $\int 20\cos({}^{4\pi t}\!/_{365})dt = 20\left({}^{365}\!/_{4\pi}\right)\sin({}^{4\pi t}\!/_{365})$ seconds or about $9.5\sin({}^{4\pi t}\!/_{365})$ minutes.

Adding these two functions gives a graph which quite closely matches those used by sundial enthusiasts (see here for example).

Seasonal Variation in Sunlit Fraction of the Day

Consider a system of coordinates centred on the Earth with Z-axis perpendicular to the Earth's orbital plane and X-axis being the intersection of equatorial and orbital planes. The Earth's spin axis is in the direction of $\langle 0, \sin\varepsilon, \cos\varepsilon \rangle$ where ε is the (constant) angle between the two planes (which is generally known as the "obliquity of the ecliptic" - and often denoted by ϕ but we'll keep that for a polar angle in spherical coordinates).

In this system, the Earth's orbital radius vector is in the XY-plane and so is given by $\langle R\cos\Theta, R\sin\Theta, 0 \rangle$ (with $\Theta = \frac{2\pi T}{365}$ where T is the number of days since the equinox).
The angle between the radius and spin vectors is thus given by
$\alpha = \arccos(\langle \cos\Theta, \sin\Theta, 0 \rangle$ [redacted] $, \sin\varepsilon, \cos\varepsilon \rangle) = \arccos(\sin\Theta \sin\varepsilon)$
This angle between the pole star and the sun varies over the year from $\frac{\pi}{2}$ at the equinoxes down to $\frac{\pi}{2} - \varepsilon$ in the Northern summer and up to $\frac{\pi}{2} + \varepsilon$ in the Northern winter

Now, for any particular day and location, consider the system of spherical coordinates rotating with the Earth, with z-axis being the earth's polar axis and the $\theta = 0$ or xz-plane being the plane containing the z-axis and the radius from the centre of the Earth through the observer (which will contain the sun at true solar noon). The direction of that radius (which points straight up relative to the observer) will then be given for an observer at latitude λ by $\langle \cos\lambda, 0, \sin\lambda \rangle$, and an object will be visible above the horizon when its direction from the Earth has a positive component in the vertical direction – i.e. when $0 < \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$ [redacted] $\text{os}\,\lambda, 0, \sin\lambda \rangle$. This simplifies to $\cos\theta > -\tan\lambda \cot\phi$.

In these coordinates the apparent position of the sun has $\phi = \alpha = \arccos(\sin\Theta \sin\varepsilon)$ and $\theta = \frac{2\pi t}{24}$ where *t* is the number of hours since noon.

The condition for visibility then gives $\cos\frac{2\pi t}{24} > -\tan(\lambda)\cot(\arccos(\sin(23°)\sin(\frac{2\pi T}{365})))$

So the number of hours of daylight at T days after the spring equinox at latitude λ is given by $\frac{24}{\pi}\arccos(\min(1, \max(-1, -\tan(\lambda)\cot(\arccos(\sin 23° \sin(\frac{2\pi T}{365}))))))$
(since in $[-\pi, \pi]$, $\cos\gamma > c$ is true for $-\arccos(c) < \gamma < \arccos(c)$ for $-1 \le c \le 1$, for the full cycle if *c*<-1 and never if *c*>1)